Partial Solution Set, Leon Section 5.4

5.4.2 Given $\mathbf{x} = (1,1,1,1)^T$ and $\mathbf{y} = (8,2,2,0)^T$,

1. Determine the angle θ between $\mathbf{x}$ and $\mathbf{y}$.
2. Find the vector projection $\mathbf{p}$ of $\mathbf{x}$ onto $\mathbf{y}$.
3. Verify that $\mathbf{x} - \mathbf{p}$ is orthogonal to $\mathbf{p}$.
4. Compute $\|\mathbf{x} - \mathbf{p}\|_2$, $\|\mathbf{p}\|_2$, and $\|\mathbf{x}\|_2$, and verify that the Pythagorean law holds.

Solution:

(a) $\theta = \arccos \frac{12}{2 \cdot 6\sqrt{2}} = \arccos \frac{1}{\sqrt{2}} = \frac{\pi}{4}$.

(b) The vector projection in question is $\mathbf{p} = \frac{1}{6}\mathbf{y} = (\frac{4}{3}, \frac{1}{3}, \frac{1}{3}, 0)^T$.

(d) We have $\|\mathbf{x} - \mathbf{p}\|_2 = \sqrt{2}$, $\|\mathbf{p}\|_2 = \sqrt{2}$, and $\|\mathbf{x}\|_2 = 2$. It follows that

$$\|\mathbf{x} - \mathbf{p}\|_2^2 + \|\mathbf{p}\|_2^2 = 2 + 2 = 4 = \|\mathbf{x}\|_2^2.$$

5.4.3 Let $\mathbf{w} = \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right)^T$, and use the weighted inner product $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{n} x_i y_i w_i$.

Let $\mathbf{x} = (1,1,1)^T$ and $\mathbf{y} = (-5,1,3)^T$.

1. Show that $\mathbf{x}$ and $\mathbf{y}$ are orthogonal with respect to this inner product.
2. Compute the values of $\|\mathbf{x}\|$ and $\|\mathbf{y}\|$ with respect to this inner product.

Solution:

1. Since $\langle \mathbf{x}, \mathbf{y} \rangle = \frac{-5}{4} + \frac{1}{2} + \frac{3}{4} = 0$, it follows that $\mathbf{x}$ and $\mathbf{y}$ are orthogonal.
2. Using the same weighted inner product, we find $\|\mathbf{x}\| = \sqrt{\frac{1}{4} + \frac{1}{2} + \frac{1}{4}} = 1$, and $\|\mathbf{y}\| = \sqrt{\frac{25}{4} + \frac{1}{2} + \frac{9}{4}} = 3$.

5.4.4 Let $A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & 2 \\ 3 & 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -4 & 1 & 1 \\ -3 & 3 & 2 \\ 1 & -2 & 2 \end{bmatrix}$, and determine the value of each of the following:

(a) $\langle A, B \rangle = \sum_{i=1}^{3} \sum_{j=1}^{3} a_{ij} b_{ij} = 0$.

(b) $\|A\|_F = \left(\sum_{i=1}^{3} \sum_{j=1}^{3} a_{ij}^2 \right)^{1/2} = 5$.

5.4.7b Using the inner product $\langle f, g\rangle = \int_0^1 f(x)g(x)dx$,

$$\begin{aligned}
\langle x, \sin \pi x\rangle &= \int_0^1 x \sin \pi x dx \\
&= \frac{-x}{\pi} \cos \pi x \Big|_0^1 + \frac{1}{\pi} \int_0^1 \cos \pi x dx \\
&= (\frac{-x}{\pi} \cos \pi x + \frac{1}{\pi^2} \sin \pi x \Big|_0^1 \\
&= (\frac{-1}{\pi}(-1-0) + \frac{1}{\pi^2}(0-0) \\
&= \frac{1}{\pi}
\end{aligned}$$

5.4.8 In $C[0, 1]$, with inner product defined by $\langle f, g\rangle = \int_0^1 f(x)g(x)dx$, consider the vectors 1 and x.

1. Find the angle θ between 1 and x.
   Solution: The angle in question is given by
   $$\theta = \arccos \frac{\langle 1, x\rangle}{\|1\|\|x\|} = \arccos \left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}.$$
2. Determine the vector projection $\mathbf{p}$ of 1 onto x, and verify that $1 - \mathbf{p}$ is orthogonal to $\mathbf{p}$.
   Solution: The vector projection is given by $\mathbf{p} = \dfrac{\langle 1, x\rangle}{\langle x, x\rangle} x = \dfrac{3}{2}x$. It follows that $1 - \mathbf{p} = 1 - \frac{3}{2}x$, and orthogonality is verified by calculating
   $$\langle \mathbf{p}, 1 - \mathbf{p}\rangle = \int_0^1 \left(\frac{3}{2}x - \frac{9}{4}x^2\right) dx = 0.$$
3. Compute $\|1 - \mathbf{p}\|$, $\|\mathbf{p}\|$, and $\|1\|$, and verify that the Pythagorean law holds.
   Solution: The respective norms are $\|1 - \mathbf{p}\| = 1/2$, $\|\mathbf{p}\| = \sqrt{3}/2$, and $\|1\| = 1$; as anticipated, $\|1 - \mathbf{p}\|^2 + \|\mathbf{p}\|^2 = 1/4 + 3/4 = 1 = \|1\|^2$.

5.4.10 Show that the functions x and x^2 are orthogonal in P_5 with respect to the inner product defined by $\langle p, q\rangle = \sum_{i=1}^{5} p(x_i)q(x_i)$, using $x_i = (i - 3)/2$ for $i = 1, \ldots, 5$.

Solution: It is straightforward to find that $\langle x, x^2\rangle = -1 - \frac{1}{8} + 0 + \frac{1}{8} + 1 = 0$.

5.4.19 Let $\mathbf{x} \in \mathbf{R}^n$. Show that $||\mathbf{x}||_\infty \leq ||\mathbf{x}||_2$.

Solution: It suffices to show that $||\mathbf{x}||_\infty^2 \leq ||\mathbf{x}||_2^2$:

$$||\mathbf{x}||_\infty^2 = \max_i |x_i| = \max_i \sqrt{x_i^2} \leq \sqrt{\sum_i x_i^2} = ||\mathbf{x}||_2^2.$$

5.4.20 Let $\mathbf{x} \in \mathbf{R}^2$. Show that $||\mathbf{x}||_2 \leq ||\mathbf{x}||_1$.

Solution: Let $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$. We show that $||\mathbf{x}||_2^2 \leq ||\mathbf{x}||_1^2$:

$$\begin{aligned} ||\mathbf{x}||_2^2 &= x_1^2 + x_2^2 \\ &\leq x_1^2 + 2|x_1||x_2| + x_2^2 \\ &= ||\mathbf{x}||_1^2, \end{aligned}$$

and we're done. $\square$

MA/Ra November 4, 2004